ELSEVIER

Journal of Geometry and Physics 45 (2003) 231–251

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Hyphs and the Ashtekar–Lewandowski measure

Christian Fleischhack*

[a] *Mathematisches Institut, Universität Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany*
[b] *Institut für Theoretische Physik, Universität Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany*
[c] *Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstrasse 22-26, 04103 Leipzig, Germany*

Received 12 January 2000

Abstract

Properties of the space $\bar{\mathcal{A}}$ of generalized connections in the Ashtekar framework are investigated. First a construction method for new connections is given. The new parallel transports differ from the original ones only along paths that pass an initial segment of a fixed path. This is closely related to a new notion of path independence. Although we do not restrict ourselves to the immersive smooth or analytical case, any finite set of paths depends on a finite set of independent paths, a so-called hyph. This generalizes the well-known directedness of the set of smooth webs and that of analytical graphs. Due to these propositions, on the one hand, the projections from $\bar{\mathcal{A}}$ to the lattice gauge theories are surjective and open. On the other hand, an induced Haar measure can be defined for every compact structure group irrespective of the used smoothness category for the paths.

MSC: Primary: 81T13; Secondary: 28C20; 53C05; 58D20

PACS: 11.15Tk

Subj. Class: Differential geometry; General relativity

Keywords: Graph; Path; Hyph; Ashtekar connections; Measure

1. Introduction

One of the recent approaches to the quantization of gauge theories, in particular of gravity, is the investigation of generalized connections introduced by Ashtekar et al. in a series of papers, see, e.g., [1–3]. Mathematically, there are two main ideas: First, every

* Correspondence address: Institut für Theoretische Physik, Universität Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany.
E-mail addresses: christian.fleischhack@itp.uni-leipzig.de, christian.fleischhack@mis.mpg.de (C. Fleischhack).

PII: S0393-0440(00)00036-X

classical (i.e. smooth) connection is uniquely determined by its parallel transports. These are certain elements of the structure group that are in a certain sense smoothly assigned to each path in the (space–time) manifold and that respect the concatenation of paths. Second, quantization here means path integral quantization. Thus, forget — as suggested by the Wiener or Feynman path integral — the smoothness of the connections being the configuration variables. Altogether, a generalized connection is simply defined to be a homomorphism from the groupoid of paths to the structure group.

At first glance this definition seems to be very rigid. But, is there a canonical choice for the groupoid $\mathcal{P}$ of paths? Do we want to restrict ourselves to piecewise analytic or immersive smooth paths? When shall two paths be equivalent? There are lots of "optimal" choices depending on the concrete problem being under consideration. For instance, for technical reasons piecewise analyticity is beautiful. In this case it is, in particular, impossible that two paths (maps from [0,1] to the manifold M) have infinitely many intersection points provided they do not coincide along a whole interval. However, since one of the most important features of gravity is the diffeomorphism invariance, one should admit at least smooth paths. Otherwise, a diffeomorphism will no longer be a map in $\mathcal{P}$. On the other hand, paths that are equal up to the parametrization, i.e., up to a map between their domains [0,1], should be equivalent. But, which maps from [0,1] onto itself are reparametrizations? As well, $\gamma \circ \gamma^{-1}$ are said to be equal to the trivial path in the initial point of the path γ. This is suggested by the homomorphy property $h_A(\gamma \circ \gamma^{-1}) = h_A(\gamma) h_A(\gamma)^{-1} = e_{\mathbf{G}}$ of the parallel transports. What are the other purely algebraic relations that h_A has to fulfill?

As just indicated, two different definitions are on the market for a couple of years. Originally, Ashtekar and Lewandowski had used the piecewise analyticity [2], and later on, Baez and Sawin [5] extended their results to the smooth category. Recently, in another paper [6] we considered a more general case. At the beginning, we only fixed the smoothness category $C^r, r \in \mathbb{N}^+ \cup \{\infty\} \cup \{\omega\}$, and decided whether we consider only piecewise immersed paths or not. Furthermore, we proposed two definitions for the equivalence of paths. The first one was — in a certain sense — the minimal one: it identifies $\gamma \circ \gamma^{-1}$ with the trivial path as well as reparametrized path. The second one identifies in the immersive case paths that are equal when parametrized w.r.t. the arc length. The main goal of our paper is a preliminary discussion for which results are insensitive to the chosen smoothness conditions and which are not.

Foremost, can an induced Haar measure be defined on the space $\bar{\mathcal{A}}$ of generalized connections in the general case? It is well known that this is indeed possible in the analytic case using graphs [2] and in the smooth case using webs [5]. What common ideas of these cases can be reused for our problem? Looking at the definition $\bar{\mathcal{A}}_{(r=\omega)} := \varprojlim_{\Gamma} \bar{\mathcal{A}}_{\Gamma}$ and $\bar{\mathcal{A}}_{\mathrm{web}} := \varprojlim_{w} \bar{\mathcal{A}}_{w}$ we see that the label sets $\{\Gamma\}$ and $\{w\}$ of the projective limit are in both cases not only projective systems, but also directed systems. This means that, e.g., for every two graphs there is a third graph such that every path in one of the first two graphs is a product of paths (or their inverses) in the third graph. The analogous result holds for the webs. In the analytical case this can be seen very easily [2], for the smooth one we refer to the paper by Baez and Sawin [5]. In [6] we defined $\bar{\mathcal{A}}$ in general by $\bar{\mathcal{A}}_{(r)} := \varprojlim_{\Gamma} \bar{\mathcal{A}}_{\Gamma}$ whereas, of course, here the graphs are in the smoothness category C^r. This definition has the drawback that the projective label set $\{\Gamma\}$ is no longer directed. But, nevertheless, note that we have shown [6] in the immersive smooth category that $\varprojlim_w \bar{\mathcal{A}}_w$ and $\bar{\mathcal{A}}_{(\infty)} = \varprojlim_\Gamma \bar{\mathcal{A}}_\Gamma$ are homeomorphic. Hence we can hope to find another appropriate label set for the case of arbitrary smoothness that generalizes the notion of webs and that gives a definition of the space of generalized connections which is equivalent to that using graphs.

In the first step we will investigate a condition for the independence of paths. When can one assign parallel transports to paths independently? As we will see, a finite set $\{\gamma_i\}$ of paths is already independent when every path γ_i contains a point v_i such that one of the subpaths of γ_i starting in v_i is non-equivalent to every subpath of the γ_j with $j < i$. Sets of paths fulfilling this condition will be called hyph. Obviously, the edges of a graph are a hyph as well as the curves of a web. The crucial point is now: For every two hyphs there is a hyph containing them. In other words, the set of hyphs is directed as the set of graphs ($r = \omega$) and that of webs ($r = \infty$). This ensures the existence of an induced Haar measure in $\bar{\mathcal{A}}_{(r)}$ for arbitrary r. Moreover, as a by-product we get an explicit construction for connections that differ from a given one only along paths that are not independent of an arbitrary, but fixed path. This immediately leads to the surjectivity of the projections π_Γ from the continuum to the lattice theory as well as that of π_w and π_v projecting to the webs and hyphs, respectively. Furthermore, we prove that π_Γ is open. In Section 6 we extend the definition of the Ashtekar–Lewandowski measure to arbitrary smoothness categories. Finally, we discuss in which cases the regular connections form a dense subset in $\bar{\mathcal{A}}_{(r)}$.

2. Notations

In this section we shall recall the basic definitions and notations introduced in [6]. For further, detailed information we refer the reader to that paper.

Let there be given a finite-dimensional, but at least two-dimensional manifold M and a (not necessarily compact) Lie group $\mathbf{G}$. Furthermore, we fix an $r \in \mathbb{N}^+ \cup \{\infty\} \cup \{\omega\}$ and decide whether we work in the category of piecewise immersive maps or not. In the following we will usually say simply C^r referring to these choices.

A path is a piecewise C^r-map from [0,1] to the manifold M. A graph consists of finitely many non-self-intersecting edges whose interiors are disjoint and contain no vertex. Paths in graphs are called simple, and finite products of simple paths are called finite paths. Two finite paths are said to be equivalent if they coincide up to piecewise C^r-reparametrizations or canceling or inserting retracings $\delta \circ \delta^{-1}$. The set of (equivalence classes of) finite paths is denoted by $\mathcal{P}$. In what follows, we say simply "path" instead of "finite path" and "graph" instead of "connected graph".

A generalized connection $\bar{A} \in \bar{\mathcal{A}}$ is a homomorphism $h_{\bar{A}} : \mathcal{P} \to \mathbf{G}$. For every graph with edges $e_i \in \mathbf{E}(\Gamma)$ and vertices $v_j \in \mathbf{V}(\Gamma)$ define the projections

$$\begin{aligned} \pi_\Gamma : \quad \bar{\mathcal{A}} &\to \bar{\mathcal{A}}_\Gamma \equiv \mathbf{G}^{\#\mathbf{E}(\Gamma)}, \\ \bar{A} &\mapsto (h_{\bar{A}}(e_1), \ldots, h_{\bar{A}}(e_{\#\mathbf{E}(\Gamma)})) \end{aligned}$$

to the lattice gauge theory. The topology on $\bar{\mathcal{A}}$ is induced using all the π_Γ by the topology of each $\mathbf{G}^{\#\mathbf{E}(\Gamma)}$.

3. A construction method for new connections

Note that in this section we mean by “path” usually not an equivalence class of paths, but a “genuine” path.

The main goal of this section is to provide a method for constructing a connection $\bar{A}$ that only minimally, but significantly differs from a given $\bar{A}'$. In detail, we want to define a new connection whose parallel transport along a given path e takes a given group element g, but has the same parallel transports as the older one along the other paths. However, this is obviously impossible, because the parallel transports have to obey the homomorphy rule. How can we find the way out? The idea goes as follows: The only condition a connection has to fulfill as a map from $\mathcal{P}$ to $\mathbf{G}$ is indeed the homomorphy property. Therefore, it should be possible to leave the parallel transports at least along those paths untouched that do not pass any subpath of our given path e. Since the generalized connections need not fulfill any continuity condition it does not matter “where” in e the modification should be placed, e.g., whether in the first half or the second or perhaps in the initial point. Since we are looking for minimal variation we try to place the modification into one single point, say, the initial point $e(0)$. This way all paths that do not pass $e(0)$ can keep their parallel transports. This is even true for those paths that though start (or end) in the point $e(0)$, but start (or end) in “another direction” as $e(0)$ does. Hence, we are now left with those paths that pass an initial path of e. There we really have to change the parallel transports — we simply multiply the corresponding factor that changes $h_{\bar{A}}(e)$ to g from the left (or its inverse from the right) to the transport of every path that starts (inversely) as e. Using a certain decomposition of an arbitrary path we get the desired construction method.

3.1. Hyphs

Before we state and prove the theorem we still need two crucial definitions and a decomposition lemma.

Definition 3.1. Let $\gamma_1, \gamma_2 \in \mathcal{P}$. We say that γ_1 and γ_2 have the same initial segment (shortly: $\gamma_1 \uparrow\uparrow \gamma_2$) iff there are non-trivial initial paths γ_1' and γ_2' of γ_1 and γ_2, respectively, that coincide up to the parametrization. We say analogously that the final segment of γ_1 coincides with the initial segment of γ_2 (shortly: $\gamma_1 \downarrow\uparrow \gamma_2$) iff $\gamma_1^{-1} \uparrow\uparrow \gamma_2$. Now the definition of $\gamma_1 \uparrow\downarrow \gamma_2$ and $\gamma_1 \downarrow\downarrow \gamma_2$ should be clear. Iff the corresponding relations are not fulfilled, we write $\gamma_1 \not\uparrow\!\not\uparrow \gamma_2$, etc.

$\gamma^{\tau,+}$ is the subpath of γ that corresponds to $\gamma|_{[\tau,1]}$; $\gamma^{\tau,-}$ that for $\gamma|_{[0,\tau]}$. Analogously, $\delta^{x,+}$ is the subpath of δ starting in x supposed $x \in \operatorname{im} \delta$. (See also [6].)

Definition 3.2. Let γ and $\delta_i, i \in I$, be the paths without self-intersections. γ is called *independent* of $D := \{\delta_i | i \in I\}$ iff

- there is a $\tau \in [0, 1)$ with $\gamma^{\tau,+} \not\uparrow\!\not\uparrow \delta_i^{\gamma(\tau),+}$ and $\gamma^{\tau,+} \not\uparrow\!\not\downarrow \delta_i^{\gamma(\tau),-}$ for all $i \in I$ or
- there is a $\tau \in (0, 1]$ with $\gamma^{\tau,-} \not\downarrow\!\not\uparrow \delta_i^{\gamma(\tau),+}$ and $\gamma^{\tau,-} \not\downarrow\!\not\downarrow \delta_i^{\gamma(\tau),-}$ for all $i \in I$.

(If $\gamma(\tau)$ should not be contained in $\operatorname{im}\delta$ then the corresponding relation $\gamma^{\tau,+}\not\uparrow\uparrow\delta_i^{\gamma(\tau),+}$, etc., is defined to be fulfilled.) The point $\gamma(\tau)$ is then usually called *free point* of γ.

A finite set $D = \{\delta_i\}$ of paths without self-intersections is called *hyph* or *moderately independent* iff δ_i is independent of $D_i = \{\delta_j \mid j < i\}$.

Lemma 3.3. *Let $\gamma \in \mathcal{P}$ and $x \in M$. Then $\gamma^{-1}(\{x\})$ is a union of at most finitely many isolated points and finitely many closed intervals in* [0,1].

Proof. Let γ be (up to the parametrization) equal $\prod \gamma_i'$ with simple $\gamma_i' \in \mathcal{P}$. Since any γ_i' equals (up to the parametrization) a finite product of edges in graphs and of trivial paths, this is also true for γ itself. Obviously, we can even assume w.l.o.g. that $\gamma = \prod \gamma_i$ with γ_i being edges in graphs or trivial paths. (Thus, the manner of writing brackets in $\prod \gamma_i$ does not matter.)

The assertion of the lemma is obviously true for any γ_i because an edge in a graph has just been defined as non-self-intersecting and $\gamma_i^{-1}(\{x\})$ is in the case of a trivial path either equal $\emptyset$ or [0,1].

The case of a general γ is now clear. □

Corollary 3.4. *Let $x \in M$ be a point. Any $\gamma \in \mathcal{P}$ can be written* (*up to parametrization*) *as a product $\prod \gamma_i$ with $\gamma_i \in \mathcal{P}$, such that*

- $\operatorname{int}\gamma_i \cap \{x\} = \emptyset$ *or*
- $\operatorname{int}\gamma_i = \{x\}$.

Proof. Mark on [0,1] the end points of the closed intervals and the isolated points of $\gamma^{-1}(\{x\})$ outside these intervals. We get finitely many intervals on [0,1]. Each one corresponds to a subpath γ_i of γ. Obviously, $\prod \gamma_i$ is the desired decomposition of γ. □

3.2. The construction

Now we state the construction method.

Construction 3.5. Let $\bar{A} \in \bar{\mathcal{A}}$ and $e \in \mathcal{P}$ be a path without self-intersections. Furthermore, let $g \in \mathbf{G}$.

We now define $h : \mathcal{P} \to \mathbf{G}$.

- Let $\gamma \in \mathcal{P}$ be for the moment a path that does not contain the initial point $e(0)$ of e as an inner point. Explicitly we have $\operatorname{int}\gamma \cap \{e(0)\} = \emptyset$. Define

$$h(\gamma) := \begin{cases} g h_{\bar{A}}(e)^{-1} h_{\bar{A}}(\gamma) h_{\bar{A}}(e) g^{-1} & \text{for} \quad \gamma \uparrow\uparrow e \text{ and } \gamma \downarrow\uparrow e, \\ g h_{\bar{A}}(e)^{-1} h_{\bar{A}}(\gamma) & \text{for} \quad \gamma \uparrow\uparrow e \text{ and } \gamma \not\downarrow\uparrow e, \\ h_{\bar{A}}(\gamma) h_{\bar{A}}(e) g^{-1} & \text{for} \quad \gamma \not\uparrow\uparrow e \text{ and } \gamma \downarrow\uparrow e, \\ h_{\bar{A}}(\gamma) & \text{else.} \end{cases}$$

- For every trivial path γ set $h(\gamma) = e_{\mathbf{G}}$.
- Now, let $\gamma \in \mathcal{P}$ be an arbitrary path. Decompose γ into a finite product $\prod \gamma_i$ due to Corollary 3.4 such that not any γ_i contains the point $e(0)$ in the interior, supposed γ_i is not trivial. Here, set $h(\gamma) := \prod h(\gamma_i)$.

Theorem 3.6. *The map* $h : \mathcal{P} \to \mathbf{G}$ *from Construction* 3.5 *is for all* $\bar{A}$, e *and* g *a homomorphism, i.e., corresponds to a connection* $\bar{A}' \in \bar{\mathcal{A}}$.

Here, $\mathcal{P}$ is the set of all *equivalence classes* of paths.

Proof.

1. h is a well-defined mapping from $\mathcal{P}$ to $\mathbf{G}$.
   - Obviously, $h(\gamma') = h(\gamma'')$ if γ' and γ'' coincide up to the parametrization. Thus, we can drop the brackets in the following when we construct multiple products of paths.
   - Now, we show $h(\delta' \circ \delta'') = h(\delta' \circ \delta \circ \delta^{-1} \circ \delta'')$.
     Decompose δ', δ'' and δ due to Corollary 3.4.
     - $\delta(0) \neq e(0), \delta(1) \neq e(0)$ and $e(0) \in \operatorname{im} \delta$. Then the decomposition of $\delta' \circ \delta''$ is equal to $(\prod_{i=1}^{I'-1} \delta_i')\gamma_*'''(\prod_{i=2}^{I''} \delta_i'')$ setting $\gamma_*''' := \delta_{I'}'\delta_1''$. The decomposition of $\delta' \circ \delta \circ \delta^{-1} \circ \delta''$ is
     $$\left(\prod_{i=1}^{I'-1} \delta_i'\right) \gamma_*' \left(\prod_{i=2}^{I-1} \delta_i\right) \gamma_* \left(\prod_{i=I-1}^{2} \delta_i^{-1}\right) \gamma_*'' \left(\prod_{i=2}^{I''} \delta_i''\right)$$
     with $\gamma_*' := \delta_{I'}'\delta_1$, $\gamma_* := \delta_I \delta_I^{-1}$ and $\gamma_*'' := \delta_1^{-1}\delta_1''$. (In the third product the index decreases.)
     A simple calculation shows that the definition above indeed yields the same parallel transport for both paths.
     - The other cases can be proven completely analogously.
   - We have as well $h(\delta' \circ \delta \circ \delta^{-1}) = h(\delta') = h(\delta \circ \delta^{-1} \circ \delta')$ for all δ' and δ.
   - Since equivalent paths can be transformed into each other by a finite number of just described transformations, we get the well-definedness.
2. h is a homomorphism, i.e., h corresponds to a generalized connection.
   Let γ and δ be the two paths and $\prod_{i=1}^{I} \gamma_i$ and $\prod_{j=1}^{J} \delta_j$ be their decompositions, respectively, as above. Then the decomposition of $\gamma \circ \delta$ equals $(\prod_{i=1}^{I-1} \gamma_i)\gamma_*(\prod_{j=2}^{J} \delta_j)$ with $\gamma_* := \gamma_I \delta_1$ supposed
   - $\gamma_I(1) \equiv \delta_1(0) \neq e(0)$ or
   - $\gamma_I(\tau) = e(0)$ for all τ and so does $\delta_1(\tau)$.

   Otherwise the decomposition is $(\prod_{i=1}^{I} \gamma_i)(\prod_{j=1}^{J} \delta_j)$ and the homomorphy is trivial by the above definition of h on general paths.
   In the first case we still have to prove $h(\gamma_I \circ \delta_1) = h(\gamma_I)h(\delta_1)$. But, this can be seen quickly using the homomorphy property of $h_{\bar{A}}$ and the definition above. □

Remark.

- *The theorem just proven is very well suited for the proof of the surjectivity and the openness of* $\pi_\Gamma : \bar{\mathcal{A}} \to \bar{\mathcal{A}}_\Gamma$ *(see below). In a certain sense it is a generalization of the proposition about the independence of loops in [2,8]. This says that (for compact Lie groups with* $\exp(\mathbf{g}) = \mathbf{G}$*) the holonomies along independent loops are even independent on the level of regular connections. For instance, a set of loops is independent if each loop possesses a subpath called free segment that is not passed by any other loop. The independence proposition could be proven modifying suitably a given connection along those free segments, such that the resulting holonomy becomes a certain fixed value. In our case we do no longer need the restriction to regular connections. We can instead modify a connection "pointwise", e.g., in the point* $e(0)$ *in the construction above.*
- *In the compact case we will extensively use this theorem in another paper [7] when we prove a stratification theorem for* $\bar{\mathcal{A}}$ *and* $\bar{\mathcal{A}}/\bar{\mathcal{G}}$*.*
- *The theorem is valid not only for compact, but also for arbitrary structure groups* $\mathbf{G}$*.*

3.3. Consequences

In this subsection we collect some immediate implications given by the construction above.

First we consider the case of arbitrarily many paths $e_i \in E$ that are, first, independent of the corresponding remaining paths in $E \setminus \{e_i\}$ and, second, whose end points form a finite set containing all the free points. Then the parallel transports can be chosen freely. More precisely, we have the following

Proposition 3.7. *Let* $\bar{A} \in \bar{\mathcal{A}}$ *and I be a set. Let* $E := \{e_i | i \in I\} \subseteq \mathcal{P}$ *be a set of paths that fulfill the following conditions:*

1. e_i *is a path without self-intersections for all i.*
2. $e_i \not\uparrow\!\uparrow e_j$ *for all* $i \neq j$*.*
3. $e_i \not\uparrow\!\downarrow e_j$ *for all* i, j*.*
4. *The set* $V_- := \{e_i(0) | i \in I\}$ *of all initial points is finite.*
5. $V_- \cap \operatorname{int} e_i = \emptyset$ *for all i.*

Finally, let there be a given $g_i \in \mathbf{G}$ *for all* $i \in I$*.*
Then, there exists an $\bar{A}' \in \bar{\mathcal{A}}$ *such that*

- $h_{\bar{A}'}(e_i) = g_i$ *for all* $i \in I$*.*
- $h_{\bar{A}'}(\gamma) = h_{\bar{A}}(\gamma)$ *for all* γ *that do not have a subpath* γ' *that fulfills* $\gamma' \uparrow\uparrow e_i$ *or* $\gamma' \downarrow\uparrow e_i$ *for some* $i \in I$*. Especially, this holds for all* γ *with* $\operatorname{im} \gamma \cap (\cup_{i \in I} \operatorname{int} e_i) = \emptyset$*.*

Proof. First we observe that it is impossible that $\gamma \uparrow\uparrow e_i$ and $\gamma \uparrow\uparrow e_j$ for $i \neq j$, because this would imply $e_i \uparrow\uparrow e_j$. Analogously, $\gamma \downarrow\uparrow e_i$ and $\gamma \downarrow\uparrow e_j$ is impossible for $i \neq j$. Now we define $h : \mathcal{P} \to \mathbf{G}$ as in Construction 3.5 with some modifications. Let $\gamma \in \mathcal{P}$. We decompose γ according to the (finite number of) passages of points in V_-. Then we set for every such subpath (again denoted by γ)

$$h(\gamma) := \begin{cases} g_i h_{\bar{A}}(e_i)^{-1} h_{\bar{A}}(\gamma) h_{\bar{A}}(e_j) g_j^{-1} & \text{if } \exists i : \gamma \uparrow\uparrow e_i \text{ and } \exists j : \gamma \downarrow\uparrow e_j, \\ g_i h_{\bar{A}}(e_i)^{-1} h_{\bar{A}}(\gamma) & \text{if } \exists i : \gamma \uparrow\uparrow e_i \text{ and } \forall j : \gamma \not\downarrow\uparrow e_j, \\ \qquad h_{\bar{A}}(\gamma) h_{\bar{A}}(e_j) g_j^{-1} & \text{if } \forall i : \gamma \not\uparrow\uparrow e_i \text{ and } \exists j : \gamma \downarrow\uparrow e_j, \\ \qquad h_{\bar{A}}(\gamma) & \text{else} \end{cases}$$

and extend the definition by homomorphy.

As in Theorem 3.6 one easily proves that h is a well-defined homomorphism using the observation in the beginning of the present proof. Hence, $h = h_{\bar{A}'}$ with some $\bar{A}' \in \bar{\mathcal{A}}$. Finally, one sees immediately from the definition of h that $h_{\bar{A}'}(e_i) = g_i$ for all $i \in I$ and $h_{\bar{A}'}(\gamma) = h_{\bar{A}}(\gamma)$ for all γ with the properties above. □

The preceding proposition covers both the case of webs and of graphs:

Corollary 3.8. *The assumptions of Proposition 3.7 are fulfilled if E is the set of all edges of a graph or the set of all curves of a web.*

Proof. For finite graphs the proof is trivial. Let, therefore, E be the set of all curves of a web. By definition, the conditions (1), (4) and (5) are fulfilled as one can easily check using the definition of a web (cf. [5]).

To prove (2) we assume that $e_1 \uparrow\uparrow e_2$ for certain curves $e_1, e_2 \in E$. Then we know that $e_1(0) = e_2(0) =: p_0$, i.e., e_1 and e_2 belong to one and the same tassel. Suppose now $\operatorname{im} e_1 \neq \operatorname{im} e_2$. Then there is w.l.o.g. a $p \in M$ with $p \in \operatorname{im} e_1 \setminus \operatorname{im} e_2$. Then, by the definition of a tassel, in every neighborhood of p_0 there is a $p' \in \operatorname{im} e_1 \setminus \operatorname{im} e_2$. But this is a contradiction to $e_1 \uparrow\uparrow e_2$. Hence, $\operatorname{im} e_1 = \operatorname{im} e_2$. Thus, since the e_l are paths without self-intersections, there is a homeomorphism $\Pi : [0,1] \to [0,1]$ with $e_2 = e_1 \circ \Pi$ and $\Pi(0) = 0$. Now, due to the consistent parametrization of curves of a tassel we know that there is a positive constant k with $\Pi(\tau) = k\tau$ for all $\tau \in [0,1]$. Because of $\Pi(1) = 1$, we get $k = 1$ and $\Pi = \mathrm{id}$. Thus, $e_2 = e_1$.

Finally, condition (3) is fulfilled. In fact, let $e_i \uparrow\downarrow e_j$. Then we have $e_i(0) = e_j(1)$. This is obviously impossible by the definition of tassels and webs. □

From the proof we get immediately

Corollary 3.9. *The curves of a web form a hyph.*

Proof. The free point of a curve c in the web is simply its initial point $c(0)$. □

Now, we come to the case of arbitrary independent paths leading to the hyphs themselves.

Proposition 3.10. *Let $\bar{A} \in \bar{\mathcal{A}}$ and $C \subseteq \mathcal{P}$ be a set of paths without self-intersections. Now, let $e \in \mathcal{P}$ be a path without self-intersections and $g \in \mathbf{G}$ be arbitrary. Furthermore, suppose that e is independent of C.*

Then there is an $\bar{A}' \in \bar{\mathcal{A}}$ *such that*

- $h_{\bar{A}'}(e) = g$;
- $h_{\bar{A}'}(c) = h_{\bar{A}}(c)$ *for all* $c \in C$.

Proof. Due to the independence of e w.r.t. C, we have $e \sim e^{\tau,-} \circ e^{\tau,+}$ for some $\tau \in [0, 1]$, such that, w.l.o.g., $e^+ := e^{\tau,+}$ is a non-trivial path such that for all subpaths c' of all the $c \in C$ we have $e^+ \not\uparrow\!\!\uparrow c'$ and $e^+ \not\uparrow\!\!\downarrow c'$. (If $\tau = 0$ we defined $e^{\tau,-}$ to be the trivial path and, analogously, $e^{\tau,+}$ for $\tau = 1$.) Analogously to Proposition 3.7 above there is now an $\bar{A}' \in \bar{\mathcal{A}}$ such that with $e^- := e^{\tau,-}$

- $h_{\bar{A}'}(e^+) = (h_{\bar{A}}(e^-))^{-1} g$;
- $h_{\bar{A}'}(c) = h_{\bar{A}}(c)$ for all c;
- $h_{\bar{A}'}(e^-) = h_{\bar{A}}(e^-)$.

The last line follows, because e is a path without self-intersections, i.e., there cannot exist a subpath e' of e^- that is $\uparrow\uparrow$ or $\downarrow\uparrow$ to e^+. Finally, we have $h_{\bar{A}'}(e) = h_{\bar{A}'}(e^-) h_{\bar{A}'}(e^+) = g$. □

Corollary 3.11. *Let* $\bar{A} \in \bar{\mathcal{A}}$ *be a generalized connection and* $\upsilon = \{e_1, \ldots, e_Y\} \subseteq \mathcal{P}$ *be a hyph. Furthermore, let* $g_i \in \mathbf{G}$, $i = 1, \ldots, Y$ *be arbitrary. Then there is a connection* $\bar{A}' \in \bar{\mathcal{A}}$ *such that* $h_{\bar{A}'}(e_i) = g_i$ *for all i.*

Proof. Use inductively the preceding proposition. Let $\bar{A}_0 := \bar{A}$. Then for all i choose an $\bar{A}_i$ such that $h_{\bar{A}_i}(e_i) = g_i$ and $h_{\bar{A}_i}(e_j) = h_{\bar{A}_{i-1}}(e_j)$ for all $j < i$ using the assumed independence of e_i w.r.t. $\{e_j | j < i\}$. Finally, set $\bar{A}' := \bar{A}_Y$. $\bar{A}'$ has now the desired property. □

3.4. *Surjectivity*

Proposition 3.12.

$\pi_\Gamma : \bar{\mathcal{A}} \to \bar{\mathcal{A}}_\Gamma$ *is surjective for all graphs* Γ.
$\pi_w : \bar{\mathcal{A}} \to \bar{\mathcal{A}}_w$ *is surjective for all webs* w.
$\pi_\upsilon : \bar{\mathcal{A}} \to \bar{\mathcal{A}}_\upsilon$ *is surjective for all hyphs* υ.

π_υ *is simply the map* $\bar{A} \mapsto (h_{\bar{A}}(e_1), \ldots, h_{\bar{A}}(e_Y)) \in \mathbf{G}^Y$ *where* e_i *are the paths in* υ.

For Lie groups with $\exp(\mathbf{g}) = \mathbf{G}$ *the surjectivity of* π_Γ *can also be proven analytically showing that even* $\pi_\Gamma|_{\mathcal{A}} : \mathcal{A} \to \bar{\mathcal{A}}_\Gamma$ *is surjective. In the case of webs one additionally needs compactness and semi-simplicity of* **G**. *But, the proof given here has the advantage that it is completely algebraic and needs no additional assumptions for* **G**. *Moreover, it uses the very constructive proposition just proved and is valid also for hyphs.*

Proof. Let $(g_1, \ldots, g_{\#\mathbf{E}(\Gamma)}) \in \mathbf{G}^{\#\mathbf{E}(\Gamma)}$ be given. Now let $\bar{A} \in \bar{\mathcal{A}}$ be the trivial connection, i.e., $h_{\bar{A}}(\gamma) = e_{\mathbf{G}}$ for all $\gamma \in \mathcal{P}$. By Proposition 3.7 and Corollary 3.8 there is an $\bar{A}' \in \bar{\mathcal{A}}$ with $h_{\bar{A}'}(e_i) = g_i$ for all $i = 1, \ldots, \#\mathbf{E}(\Gamma)$.

The proof in the case of webs is completely analogous, the proof for hyphs uses Corollary 3.11. □

3.5. Definition of $\bar{\mathcal{A}}$ using hyphs

In another paper [6] we proved that in the smooth case for a compact and semi-simple structure group $\mathbf{G}$ the spaces $\bar{\mathcal{A}}_{(\infty,+)}$ and $\bar{\mathcal{A}}_{\mathrm{web}}$ of generalized connections used here and by Baez and Sawin [5], respectively, are in fact homeomorphic. Now, we will translate that proof to the case of hyphs.

First, we define a partial ordering on the set of hyphs: $\upsilon_1 \leq \upsilon_2$ iff every $e \in \upsilon_1$ equals up to the parametrization a finite product of paths in υ_2 and their inverses. Then we can define $\bar{\mathcal{A}}_\upsilon := \mathrm{Hom}(\mathcal{P}_\upsilon, \mathbf{G})$ ($\mathcal{P}_\upsilon$ being the subgroupoid of $\mathcal{P}$ generated by υ) and

$$\begin{array}{rccc} \pi^{\upsilon_2}_{\upsilon_1}: & \bar{\mathcal{A}}_{\upsilon_2} & \to & \bar{\mathcal{A}}_{\upsilon_1}, \\ & h & \mapsto & h|_{\mathcal{P}_{\upsilon_1}} \end{array}$$

for $\upsilon_1 \leq \upsilon_2$. We topologize $\bar{\mathcal{A}}_\upsilon$ identifying it with $\mathbf{G}^{\#\upsilon}$. Obviously $\pi^{\upsilon_2}_{\upsilon_1}$ is always continuous, surjective and open. So we can define $\bar{\mathcal{A}}_{\mathrm{hyph}} := \varprojlim_\upsilon \bar{\mathcal{A}}_\upsilon$ as the space of generalized connections with the canonical projections

$$\begin{array}{rccc} \pi_\upsilon: & \bar{\mathcal{A}}_{\mathrm{hyph}} & \to & \bar{\mathcal{A}}_\upsilon, \\ & (h_{\upsilon'})_{\upsilon'} & \mapsto & h_\upsilon. \end{array}$$

Using the surjectivity of π_υ we get

Proposition 3.13. *$\bar{\mathcal{A}}_{\mathrm{hyph}}$ and $\bar{\mathcal{A}}$ are homeomorphic in every smoothness category.*

The proof is almost literally the same as for $\bar{\mathcal{A}}_{\mathrm{web}}$ and $\bar{\mathcal{A}}_{(\infty,+)}$ in [6] and is therefore dropped here.

4. Directedness of the set of hyphs

In this section we will prove the following

Theorem 4.1. *The set of all hyphs is directed.*

This assertion follows immediately from the more general

Proposition 4.2. *Let $C \subseteq \mathcal{P}$ be a finite set of paths without self-intersections. Then there is a hyph υ, such that every $c \in C$ equals up to the parametrization a finite product of paths (and their inverses) in υ.*

Consequently, for no $c \in C$ there is a path occurring twice in the product for c.

We will prove this theorem using induction on the number of paths in C. If a path $c \in C$ would be independent of the complement $C \setminus \{c\}$, there will be no problems. Therefore, we first consider the other case.

4.1. Non-independent paths

In the following we often decompose paths without self-intersections according to a finite set P of points in the manifold M. This means, given some path e we construct non-trivial subpaths e_i such that every e_i starts and ends in P or $e(0)$ or $e(1)$. We obviously need only finitely many e_i and get $e \sim \prod e_i$.

Lemma 4.3. *Let e and c_j, $j \in J$, be finitely many paths without self-intersections, such that e is not independent of $C := \{c_j \mid j \in J\}$.*

Then there are $\tau_i \in [0,1]$, $i = 0, \ldots, I$, with $\tau_0 = 0$ and $\tau_I = 1$ such that the following holds: After decomposing every e and c_j into a product of edges $\prod_{i=0}^{I-1} e_i$ and $\prod c'_k$, respectively, according to the set $\{e(\tau_i)\}$ for every $i = 0, \ldots, I-1$, one of the following two assertions is true:

1. $e_i \uparrow\uparrow c'_k \Rightarrow e_i \sim c'_k$ *and* $e_i \uparrow\downarrow c'_k \Rightarrow e_i \sim (c'_k)^{-1}$;
2. $e_i \downarrow\uparrow c'_k \Rightarrow (e_i)^{-1} \sim c'_k$ *and* $e_i \downarrow\downarrow c'_k \Rightarrow (e_i)^{-1} \sim (c'_k)^{-1}$.

Note that here the $\sim$-sign indicates that, e.g., in the first case, e_i and c'_k are even equal up to the parametrization.

Proof. (1) Let $I_{\tau,+,j}$, $\tau \in [0,1]$, contain exactly τ itself and those $\tau' \in (\tau, 1]$ for that the subpath of e from τ to τ' is up to the parametrization equal to some subpath of c_j or c_j^{-1}. By assumption for all $\tau \in [0,1)$ there is a j with $I_{\tau,+,j} \neq \{\tau\}$.

Analogously, $I_{\tau,-,j}$, $\tau \in [0,1]$, contains exactly τ itself and those $\tau' \in [0,\tau)$ for that the subpath of e from τ' to τ is up to the parametrization equal to some subpath of c_j or c_j^{-1}. Again, by assumption for all $\tau \in (0,1]$ there is a j with $I_{\tau,-,j} \neq \{\tau\}$.

Furthermore, $I_{\tau,\pm,j}$ is everytime connected.

Now, define

$$I_{\tau,\pm} := \bigcap_{\substack{j \in J \\ I_{\tau,\pm,j} \neq \{\tau\}}} I_{\tau,\pm,j},$$

as well as $I_{0,-} := \{0\}$ and $I_{1,+} := \{1\}$.

What is the interpretation of such an $I_{\tau,\pm}$? $I_{\tau,+}$, e.g., is that interval in [0,1] starting in τ such that every subpath of c_j (or c_j^{-1}), that starts in $e(\tau)$ as e does, is even equal (up to the parametrization) to this subpath of e at least from $e(\tau)$ to $e(\tau')$ for every $\tau' \in I_{\tau,\pm}$. However, note that $I_{\tau,\pm}$ need not be a closed interval. Observe that $I_{\tau,\pm}$ is in each case (except for $I_{0,-}$ and $I_{1,+}$) an interval that contains $\{\tau\}$ as a proper subset.

(2) Now, we construct a sequence (τ_i) of numbers starting with $\tau_0 := 0$ as follows for all $i \geq 0$:

1. $\tau_{i,+} := \sup I_{\tau_i,+}$.
2. $\tau_{i+1} := \sup \{\tau \in [\tau_{i,+}, 1] \mid I_{\tau_i,+} \cap I_{\tau,-} \neq \emptyset\}$.
3. $\tau_{i+1,-}$ is some number with

- $\tau_{i,+} \leq \tau_{i+1,-} \leq \tau_{i+1}$;
- $\tau_{i+1,-} \in I_{\tau_{i+1},-}$;
- $I_{\tau_i,+} \cap I_{\tau_{i+1,-},-} \neq \emptyset$.

4. $\tau_{i+(1/2)}$ is some number in $I_{\tau_i,+} \cap I_{\tau_{i+1,-},-}$.
5. If $\tau_{i+1} = 1$ then stop the procedure.

Observe:

1. $\tau_{i,+} > \tau_i$, because $I_{\tau_i,+}$ is a non-trivial interval.
2. Since $I_{\tau_i,+} \cap I_{\tau_{i,+},-} \neq \emptyset$ (by definition of $\tau_{i,+}$), the set of all numbers τ with $I_{\tau_i,+} \cap I_{\tau,-} \neq \emptyset$ and $\tau \geq \tau_{i,+}$ is non-empty. Consequently, it has a supremum $\tau_{i+1} \geq \tau_{i,+}$.
3. By choice of τ_{i+1} as such a supremum there is a $\tau' \geq \tau_{i,+}$ with $\tau' \in I_{\tau_{i+1},-}$ and $I_{\tau_i,+} \cap I_{\tau',-} \neq \emptyset$. Choose now $\tau_{i+1,-} := \tau'$.
4. $\tau_{i+(1/2)}$ exists obviously.

Thus, the construction above is possible.

Furthermore, we have $\tau_i \leq \tau_{i+(1/2)} \leq \tau_{i,+} \leq \tau_{i+1,-} \leq \tau_{i+1}$ and $\tau_i < \tau_{i+1}$.

(3) Now, assume that there is no $N \in \mathbb{N}$ with $\tau_N = 1$. Then $(\tau_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence with values in $[0, 1)$, i.e. $\tau_i \to \tau \in (0, 1]$ for $i \to \infty$, and we have $\tau_i < \tau$ for all $i \in \mathbb{N}$.

Let $\tau' \in I_{\tau,-}$ with $\tau' < \tau$. Then there is an $n \in \mathbb{N}$ with $\tau' \leq \tau_n < \tau$. Now we have $I_{\tau_n,+} \cap I_{\tau,-} \neq \emptyset$, because, e.g., τ_n is contained in this set. But, from this we get together the step 2 of (2) above, that $\tau \leq \tau_{n+1}$. This is a contradiction to $\tau > \tau_{n+1}$. Consequently, there is an $N \in \mathbb{N}$ with $\tau_N = 1$.

(4) Now, the desired parameter values are τ_i, $\tau_{i+(1/2)}$ and $\tau_{i+1,-}$ for $i = 0, \ldots, N-1$ as well as τ_N. Divide the edges e and c_j according to the set of all those $e(\tau_{\ldots})$. We have (if two subsequent vertices $e(\tau_{\ldots})$ are equal, we drop the correspondent (trivial) subpaths $e_{\ldots}$ and $c'_{\ldots}$):

1. $e_i \uparrow\uparrow c'_k \Rightarrow e_i \sim c'_k$ and $e_i \uparrow\downarrow c'_k \Rightarrow e_i \sim (c'_k)^{-1}$;
2. $e_{i+(1/2)} \downarrow\uparrow c'_k \Rightarrow (e_{i+(1/2)})^{-1} \sim c'_k$ and $e_{i+(1/2)} \downarrow\downarrow c'_k \Rightarrow (e_{i+(1/2)})^{-1} \sim (c'_k)^{-1}$;
3. $e_{i+1,-} \downarrow\uparrow c'_k \Rightarrow (e_{i+1,-})^{-1} \sim c'_k$ and $e_{i+1,-} \downarrow\downarrow c'_k \Rightarrow (e_{i+1,-})^{-1} \sim (c'_k)^{-1}$.

We only show the first item, the other two can be proven analogously.

Let $e_i \uparrow\uparrow c'_k$. Since c'_k is a subpath of a c_j, we have $I_{\tau_i,+,j} \neq \{\tau_i\}$. From $I_{\tau_i,+,j} \supseteq I_{\tau_i,+} \supseteq [\tau_i, \tau_{i+(1/2)}]$ we get now e_i equals (up to the parametrization) a subpath of c_j starting in $e(\tau_i)$. But, since c_j has no self-intersections and is divided according to $e(\tau_i)$ and $e(\tau_{i+(1/2)})$ (and other vertices that are not contained in im e_i), we have e_i even equals c'_k up to the parametrization.

In the case $e_i \uparrow\downarrow c'_k$ we conclude analogously using $e_i \uparrow\uparrow (c'_k)^{-1}$. □

4.2. *Proof of Proposition 4.2*

Proof of Proposition 4.2.

- First of all we decompose all c_i according to the set $V := \{c_i(0)\}_i \cup \{c_i(1)\}_i$ of all end points. Thus, we get a finite set C' of paths without self-intersections, whereas every

$c \in C$ equals up to the parametrization a finite product of paths $c' \in C'$ and their inverses and where no end point of a path c' is contained in the interior of another path in C'. Consequently, we can w.l.o.g. assume that our set C in the proposition is of that type.

- Now, we consider $c_1 \in C$.
  - In the case that c_1 is already independent of $\{c_j | j > 1\}$ we need not decompose c_1; we simply set $c_{i,1} := c_i$ and $I_i := 1$ for all i.
  - In the other case we use Lemma 4.3 and get certain paths e_k (w.l.o.g. such that $c_1 \sim e_1 \circ \cdots \circ e_{I_1}$) such that every c_j is a product of the e_k (and their inverses) and such that the e_k, $k \in [1, I_1]$, are independent of the remaining paths. Now, we set $c_{1,k} := e_k$ for all $k \in [1, I_1]$. Analogously, we define $c_{i,l}$ for $i > 1$ being that e_k that (or whose inverse) is used at the lth position in the product for c_i, after we cancelled all e_k occurring in c_1, and denote the number of factors left by I_i.

    (*Example*: $c_1 = e_1 e_2 e_3, c_2 = e_1^{-1} e_4 e_3 e_5^{-1}$ and $c_3 = e_2^{-1}$. Then we have $I_1 = 3, I_2 = 2, I_3 = 0$ and $c_{1,1} = e_1, c_{1,2} = e_2, c_{1,3} = e_3, c_{2,1} = e_4$ and $c_{2,2} = e_5$.)

  Per constructionem, $c_{1,l}$ is independent of $\{c_{i,l'} | i > 1 \text{ or } l \neq l'\}$. Note, moreover, that the set of end points of the $c_{i,l}$ is again disjoint to the interiors of these paths. Finally, we set $C_1 := \{c_{i,l} | i > 1\}$.
- Now, we decompose the paths $c_{2,l} \in C_1$ (if $I_2 \neq 0$).

  We start with $c_{2,1}$. If it is not independent of the $\{c_{i,l} \in C_1 | i > 2 \text{ or } l \neq 1\}$, then decompose it again by Lemma 4.3 by certain independent paths e'_k. We get as before $c_{2,1} \sim c_{2,1,1} \circ \cdots \circ c_{2,1,I_{2,1}}$ and a certain set $C_{2,1}$ that collects all paths used for the decomposition of $c_{i,l}$ with $i > 2$. But, note that $c_{2,l}$ is *not* decomposed for $l \neq 1$ by that procedure.

  Afterwards, we decompose $c_{2,2}$ (w.r.t. $C_{2,1}$) and so on.

  Summa summarum, we get paths c_{2,l,m_l} with $c_{2,l} \sim \prod_{m_l} c_{2,l,m_l}$ and a set $C_2 := C_{2,I_2}$ collecting all the paths that $c_{i,l}$ with $i > 2$ is decomposed into, but that are not used in the decomposition of $c_{2,l}$. By the construction, c_{2,l,m_l} is independent of $\{c_{2,l',m'_{l'}} | l \neq l' \text{ or } m_l \neq m'_{l'}\} \cup C_2$.
- In the next step, we first collect all paths in C_2 that are used for the decomposition of c_3. After renumbering these paths by $c_{3,1}, \ldots, c_{3,I_3}$ we can again apply the previous step.
- Inductively, we get an ordered set

$$C^* = \{c_{N,1,1}, \ldots, c_{N,I_N,M_{N,I_N}}; \ldots; c_{2,1,1}, \ldots, c_{2,I_2,M_{2,I_2}}; c_{1,1}, \ldots, c_{1,I_1}\}$$

  of paths that is by construction moderately independent, consequently a hyph, and that admits a factorization of every $c_i \in C$ into a product of paths in C^* of the desired type. □

4.3. Open problem

In contrast to the case of graphs or webs we need for the definition of the independence in the case of hyphs an ordering among the paths collected in a hyph. Thus, it would be — at least for technical reasons — desirable to solve the following open problem: Does there exist for every given finite set C of paths a set E of strongly independent paths, such that every path in C is a product of paths in E and their inverses? Here strongly independent means that every path in E is independent of the remaining paths in E. We indicate the problems that arose when we tried to prove the following answers:

"Yes" The induction used for the proof of Proposition 4.2 cannot be reused. The problem is the following. Suppose we have decomposed the first path c_1 in C w.r.t. the remaining paths as above. Then we decompose (the subpaths of) the second path c_2 in C w.r.t. the others. Now, it is possible that vertices used in this procedure for the division of c_2 lie on c_1 again. Thus, c_1 would now be divided once more — with the effect that sometimes subpaths of c_1 are created that do not fulfill the independence condition. (Remember that independence means existence of *one* point in a path with the independence-of-germs condition above.) Hence, we have to divide the respective path again. But, now we could end up in a never-ending procedure that creates an infinite number of subpaths.

"No" It would be enough to present one counterexample. But, up to now, none of the examples we checked lead to a contradiction.

5. Openness of π_Γ

Proposition 5.1. *$\pi_\Gamma : \bar{\mathcal{A}} \to \bar{\mathcal{A}}_\Gamma$ is open for all graphs Γ.*

Proof. We have to show that $\pi_\Gamma(V)$ is open for all elements V of a basis of $\bar{\mathcal{A}}$, i.e., $\pi_\Gamma(\pi^{-1}_{\Gamma'_1}(W_1) \cap \cdots \cap \pi^{-1}_{\Gamma'_I}(W_I))$ is open for all graphs Γ'_i and all elements W_i of a basis of $\bar{\mathcal{A}}_{\Gamma'_i} = \mathbf{G}^{\#\mathbf{E}(\Gamma'_i)}$. But, a basis hereof is given by all sets of the type $W_{i,1} \times \cdots \times W_{i,\#\mathbf{E}(\Gamma'_i)}$ with open $W_{i,n_i} \subseteq \mathbf{G}$. Now we have

$$\pi_\Gamma(\pi^{-1}_{\Gamma'_1}(W_1) \cap \cdots \cap \pi^{-1}_{\Gamma'_I}(W_I)) = \pi_\Gamma\left(\bigcap_{i=1}^{I} \bigcap_{j_i=1}^{\#\boldsymbol{E}(\Gamma'_i)} \pi^{-1}_{e_{i,j_i}}(W_{i,j_i})\right).$$

(W.l.o.g. we assumed that none of the Γ'_i consists of a single vertex.)

Let us therefore prove the openness of all sets of the type

$$\pi_\Gamma\left(\bigcap_{j=1}^{J} \pi^{-1}_{c_j}(W_j)\right)$$

with edges c_j and open $W_j \subseteq \mathbf{G}$.

Let us denote the edges of Γ by e_i and set $E := \{e_i\}$ and $C := \{c_j\}$.

(1) Suppose first there is an $e \in E$ that is independent of C. Then it is obviously independent of $C \cup (\mathbf{E}(\Gamma) \setminus \{e\})$. We will show that

$$\pi_\Gamma\left(\bigcap_{j=1}^{J} \pi^{-1}_{c_j}(W_j)\right) = \pi_{\Gamma\setminus\{e\}}\left(\bigcap_{j=1}^{J} \pi^{-1}_{c_j}(W_j)\right) \times \mathbf{G}.$$

"$\subseteq$" Trivial.

"$\supseteq$" Let $(\vec{g}, g) \in \pi_{\Gamma\setminus\{e\}}(\cap_{j=1}^{J} \pi_{c_j}^{-1}(W_j)) \times \mathbf{G}$.
Hence, there is an $\bar{A} \in \cap_{j=1}^{J} \pi_{c_j}^{-1}(W_j)$
with $\pi_{\Gamma\setminus\{e\}}(\bar{A}) = \vec{g}$. Due to Proposition 3.10 there is an $\bar{A}' \in \bar{\mathcal{A}}$ fulfilling
- $h_{\bar{A}'}(e_i) = h_{\bar{A}}(e_i)$ for all $e_i \neq e$, i.e., $\vec{g} = \pi_{\Gamma\setminus\{e\}}(\bar{A}) = \pi_{\Gamma\setminus\{e\}}(\bar{A}')$;
- $h_{\bar{A}'}(c_j) = h_{\bar{A}}(c_j)$ for all $j = 1, \ldots, J$, i.e., $\bar{A}' \in \pi_{c_j}^{-1}(W_j)$ for all j;
- $h_{\bar{A}'}(e) = g$.

With this we have $\pi_\Gamma(\bar{A}') = (\pi_{\Gamma\setminus\{e\}}(\bar{A}'), \pi_e(\bar{A}')) = (\vec{g}, g)$, i.e.

$$(\vec{g}, g) \in \pi_\Gamma \left(\bigcap_{j=1}^{J} \pi_{c_j}^{-1}(W_j) \right).$$

(2) Successively applying the preceding step we get

$$\pi_\Gamma \left(\bigcap_{j=1}^{J} \pi_{c_j}^{-1}(W_j) \right) = \pi_{\Gamma_0} \left(\bigcap_{j=1}^{J} \pi_{c_j}^{-1}(W_j) \right) \times \mathbf{G}^n.$$

Here n denotes the number of edges e of Γ that are independent of C. Γ_0 denotes the graph that arises from Γ by removing all such edges.

(3) Since every edge e in Γ_0 is not independent of C, we can divide e_1 and the $c_j \in C$ as in Lemma 4.3 and get paths $e_{1,1}, \ldots, e_{1,n_1}$ and $c_{j,1}, \ldots, c_{j,m_j}$. We collect the $c_{\ldots}$ into $C_1 \subseteq \mathcal{P}$. Since e_i are edges of one and the same graph, e_i (for $i > 1$) is still not independent of C_1. We again use Lemma 4.3, now for decomposing e_2 and the paths in C_1. We get paths $e_{2,1}, \ldots, e_{2,n_2}$ and a $C_2 \subseteq \mathcal{P}$. Successively, we decompose all e_i and C_{i-1} getting e_{k,k_i} and $c'_l \in C' \subseteq \mathcal{P}$, such that for every i and k_i one of the following two assertions is true:

1. $e_{i,k_i} \uparrow\uparrow c'_l \Rightarrow e_{i,k_i} \sim c'_l$ and $e_{i,k_i} \uparrow\downarrow c'_l \Rightarrow e_{i,k_i} \sim (c'_l)^{-1}$;
2. $e_{i,k_i} \downarrow\uparrow c'_l \Rightarrow (e_{i,k_i})^{-1} \sim c'_l$ and $e_{i,k_i} \downarrow\downarrow c'_l \Rightarrow (e_{i,k_i})^{-1} \sim (c'_l)^{-1}$.

To reduce the technical efforts we first invert all e_{i,k_i} that fulfill the second assertion. Afterwards, we invert c'_l if it is equivalent to an $(e_{i,k_i})^{-1}$. This is possible, because there is at most one such edge $e_{\ldots}$.

It is clear, that e_{i,k_i} span a graph $\Gamma' \geq \Gamma_0$, and we know from the construction that no int c'_l contains a vertex of Γ'. Furthermore, every c_j is equivalent to a finite product of c'_l (or its inverse). The factors used for c_j (again denoted by c_{j,l_j}) span a graph Γ_j, as well. Thus, we have $\pi_{\Gamma_0} = \pi_{\Gamma_0}^{\Gamma'} \pi_{\Gamma'}$ and $\pi_{c_j}^{-1} = \pi_{\Gamma_j}^{-1} (\pi_{c_j}^{\Gamma_j})^{-1}$.

Finally, $(\pi_{c_j}^{\Gamma_j})^{-1}(W_j)$ is open in $\mathbf{G}^{m_j}$ by continuity, i.e., a union of sets of the type $W_{j,1} \times \cdots \times W_{j,m_j}$. Thus, $\pi_{\Gamma_0}(\cap_{j=1}^{J} \pi_{c_j}^{-1}(W_j))$ is the union of sets of the type $\pi_{\Gamma_0}^{\Gamma'} \pi_{\Gamma'}(\cap_{j=1}^{J} \cap_{l_j=1}^{m_j} \pi_{c_{j,l_j}}^{-1}(W_{j,l_j}))$.

(4) Due to the openness of $\pi_{\Gamma_0}^{\Gamma'}$ (see [6]) it is sufficient to prove the openness of $\pi_{\Gamma'}(\cap_{l=1}^{L} \pi_{c_l}^{-1}(W_l))$ whenever the following holds:

1. Γ' is a graph and $C' = \{c_l\}$ is a finite set of paths without self-intersections;
2. $\operatorname{int} c_l \cap \mathbf{V}(\Gamma') = \emptyset$;

3. $(e \uparrow\uparrow c_l \Rightarrow e \sim c_l)$ and $e \not\uparrow\!\!\downarrow c_l$ for all l and for every edge e of the graph Γ';
4. $W_l \subseteq \mathbf{G}$ is open for all l.

We will prove for non-empty left-hand side

$$\pi_{\Gamma'}\left(\bigcap_{l=1}^{L}\pi_{c_l}^{-1}(W_l)\right) = \times_{e_k\in\mathbf{E}(\Gamma')}\left(\bigcap_{c_l\in C(e_k)} W_l\right), \tag{1}$$

where $C(e_k) \subseteq C'$ contains exactly those $c_l \in C'$ that are (up to the parametrization) equal to e_k. Since the right-hand side is obviously open, the openness is proven if (1) is.

"$\subseteq$" Let $\vec{g} \in \pi_{\Gamma'}(\cap_{l=1}^{L}\pi_{c_l}^{-1}(W_l))$, i.e., there is an $\bar{A} \in \bar{\mathcal{A}}$ with $\pi_{e_k}(\bar{A}) = g_k$ for all k and $\pi_{c_l}(\bar{A}) \in W_l$ for all $c_l \in C'$. From this follows $g_k \in W_l$ for all $c_l \in C(e_k)$ and so $\vec{g} \in \times_{e_k\in\mathbf{E}(\Gamma')}(\cap_{c_l\in C(e_k)} W_l)$.

"$\supseteq$" Let $\vec{g} \in \times_{e_k\in\mathbf{E}(\Gamma')}(\cap_{c_l\in C(e_k)} W_l)$. Choose an $\bar{A}_0 \in \bar{\mathcal{A}}$ with $\pi_{c_l}(\bar{A}_0) \in W_l$ for all c_l. By assumption every e_k is independent of $C' \setminus (\cup_{k'} C(e_{k'}))$ and so by Proposition 3.10 there exists an $\bar{A} \in \bar{\mathcal{A}}$ such that
- $\pi_{e_k}(\bar{A}) = g_k$ for all k,
- $\pi_{c_l}(\bar{A}) = \pi_{c_l}(\bar{A}_0)$ for all c_l that are not equal (up to the parametrization) to an e_k.

Thus, we have $\pi_{c_l}(\bar{A}) \in W_l$ for all $c_l \in C(e_k)$. Consequently, $\vec{g} \in \pi_{\Gamma'}(\cap_{l=1}^{L} \pi_{c_l}^{-1}(W_l))$. □

6. Induced Haar measure

In this section we will show that thanks to the directedness of the set of hyphs an induced Haar measure can be defined with an arbitrary smoothness assumption for the paths. Our definition covers that of Ashtekar and Lewandowski [2] for graphs in the analytic category as well as that of Baez and Sawin [5] for webs in the smooth category.

Throughout this section, $\mathbf{G}$ is a *compact* Lie group.

6.1. Cylindrical functions

In this subsection we will investigate the algebra of continuous functions on $\bar{\mathcal{A}}$. Particularly nice is the dense subalgebra of the so-called cylindrical functions [2,3]. These are functions depending only on the parallel transports along a finite number of paths.

Definition 6.1. A function $f \in C(\bar{\mathcal{A}})$ is called a *genuine cylindrical function* on $\bar{\mathcal{A}}$ iff there is a graph Γ and a continuous function $f_\Gamma \in C(\bar{\mathcal{A}}_\Gamma)$ with $f = f_\Gamma \circ \pi_\Gamma$. The set of all genuine cylindrical functions is denoted by $\mathrm{Cyl}_0(\bar{\mathcal{A}})$.

Obviously, $\mathrm{Cyl}_0(\bar{\mathcal{A}})$ is $*$-invariant. But, since for two finite graphs there need not exist a third one containing both, the sum as well as the product of two cylindrical functions is no longer a cylindrical function, in general. Therefore, we enlarge the definition above to hyphs.

Definition 6.2. A function $f \in C(\bar{\mathcal{A}})$ is called *cylindrical function* on $\bar{\mathcal{A}}$ iff there is a hyph υ and a continuous function $f_\upsilon \in C(\bar{\mathcal{A}}_\upsilon)$ with $f = f_\upsilon \circ \pi_\upsilon$. The set of all cylindrical functions is denoted by $\mathrm{Cyl}(\bar{\mathcal{A}})$.

Lemma 6.3. $\mathrm{Cyl}(\bar{\mathcal{A}})$ *is a normed* $*$*-algebra containing* $\mathrm{Cyl}_0(\bar{\mathcal{A}})$.

Proof. $\mathrm{Cyl}(\bar{\mathcal{A}})$ is obviously closed w.r.t. scalar multiplication and involution. It remains to prove that it is closed w.r.t. addition and multiplication.

Let $f' = f'_{\upsilon'} \circ \pi_{\upsilon'}$ and $f'' = f''_{\upsilon''} \circ \pi_{\upsilon''}$. By Theorem 4.1 there is a hyph υ with $\upsilon \geq \upsilon', \upsilon''$. Thus we have $f' + f'' = f'_{\upsilon'} \circ \pi^{\upsilon}_{\upsilon'} \circ \pi_\upsilon + f''_{\upsilon''} \circ \pi^{\upsilon}_{\upsilon''} \circ \pi_\upsilon = (f'_{\upsilon'} \circ \pi^{\upsilon}_{\upsilon'} + f''_{\upsilon''} \circ \pi^{\upsilon}_{\upsilon''}) \circ \pi_\upsilon \in \mathrm{Cyl}(\bar{\mathcal{A}})$. Analogously, $f' \cdot f'' \in \mathrm{Cyl}(\bar{\mathcal{A}})$. □

Proposition 6.4. $\mathrm{Cyl}(\bar{\mathcal{A}})$ *is dense in* $C(\bar{\mathcal{A}})$.

Proof. The assertion follows from the Stone–Weierstraß theorem:

- $1 \in \mathrm{Cyl}(\bar{\mathcal{A}})$, whereas $1 : \bar{\mathcal{A}} \to \mathbb{C}$ is the function $1(\bar{A}) := 1$.
- $\mathrm{Cyl}(\bar{\mathcal{A}})$ separates the points of $\bar{\mathcal{A}}$. (We prove even $\mathrm{Cyl}_0(\bar{\mathcal{A}})$ separates the points of $\bar{\mathcal{A}}$.) Let $\bar{A}_1, \bar{A}_2 \in \bar{\mathcal{A}}$ with $\bar{A}_1 \neq \bar{A}_2$. Thus, there is a graph Γ with $\pi_\Gamma(\bar{A}_1) \neq \pi_\Gamma(\bar{A}_2)$. Since $\bar{\mathcal{A}}_\Gamma \equiv \mathbf{G}^{\#\mathbf{E}(\Gamma)}$ is a manifold, hence completely regular, the continuous functions on $\bar{\mathcal{A}}_\Gamma$ separate the points of $\bar{\mathcal{A}}_\Gamma$ [9]. This means there is an $f_\Gamma \in C(\bar{\mathcal{A}}_\Gamma)$ with $f_\Gamma(\pi_\Gamma(\bar{A}_1)) \neq f_\Gamma(\pi_\Gamma(\bar{A}_2))$.

  Due to $f_\Gamma \circ \pi_\Gamma \in \mathrm{Cyl}(\bar{\mathcal{A}})$, $\mathrm{Cyl}(\bar{\mathcal{A}})$ separates the points of $\bar{\mathcal{A}}$. □

6.2. The induced Haar measure on $\bar{\mathcal{A}}$

According to the Riesz–Markow theorem measures on a compact Hausdorff space are in one-to-one correspondence to linear, continuous, positive functionals on the functional algebra over that space. We get

Proposition 6.5. *For every linear, continuous, positive functional F on $C(\bar{\mathcal{A}})$ there is a unique regular Borel measure μ on $\bar{\mathcal{A}}$, such that*

$$\begin{array}{rlll} F: & C(\bar{\mathcal{A}}) & \to & \mathbb{C}, \\ & f & \mapsto & \int_{\bar{\mathcal{A}}} f \, \mathrm{d}\mu. \end{array}$$

Due to the denseness of $\mathrm{Cyl}(\bar{\mathcal{A}})$ in $C(\bar{\mathcal{A}})$, it is sufficient to define an appropriate functional on $\mathrm{Cyl}(\bar{\mathcal{A}})$ and to extend this continuously to a functional on $C(\bar{\mathcal{A}})$. One possibility is to replace the integration of functions $f_\upsilon \circ \pi_\upsilon$ over $\bar{\mathcal{A}}$ by the integration of f_υ over $\bar{\mathcal{A}}_\upsilon = \mathbf{G}^{\#\upsilon}$. But, on $\mathbf{G}^{\#\upsilon}$ there is a "canonical" measure, the Haar measure. Hence, we define (cf. [2]):

Definition 6.6. Let $f \in \mathrm{Cyl}(\bar{\mathcal{A}})$. Define $F_0(f) := \int_{\bar{\mathcal{A}}_\upsilon} f_\upsilon \, \mathrm{d}\mu_{\mathrm{Haar}}$ if $f_\upsilon \circ \pi_\upsilon = f$, and extend F_0 continuously to a functional F on $C(\bar{\mathcal{A}})$.

Proposition 6.7. $F : C(\bar{\mathcal{A}}) \to \mathbb{C}$ *is a well-defined, linear, continuous, positive functional on* $C(\bar{\mathcal{A}})$.

Furthermore, there is a unique Borel measure μ_0 *on* $\bar{\mathcal{A}}$ *with* $F(f) = \int_{\bar{\mathcal{A}}} f \,\mathrm{d}\mu_0$ *for all* $f \in C(\bar{\mathcal{A}})$.

Definition 6.8. The measure μ_0 of the preceding proposition is called *induced Haar measure* or *Ashtekar–Lewandowski measure* on $\bar{\mathcal{A}}$.

Proof.

- F_0 is well defined.

  Let f be cylindrical w.r.t. υ' and υ''. Then f is again cylindrical w.r.t. υ, if υ is some hyph containing υ' and υ''. The existence of such a υ is guaranteed by Theorem 4.1. Hence, it is sufficient to prove $\int_{\bar{\mathcal{A}}_\upsilon} f_\upsilon \,\mathrm{d}\mu_{\mathrm{Haar}} = \int_{\bar{\mathcal{A}}_{\upsilon'}} f_{\upsilon'} \,\mathrm{d}\mu_{\mathrm{Haar}}$ for all $\upsilon \geq \upsilon'$.

  Let now $\upsilon \geq \upsilon'$. Then every path e'_i of υ' can be written as a product $\prod_{k_i} e^{\pm 1}_{j(k_i,i)}$ of paths in υ (and their inverses). By the moderate independence of hyphs there is a path $e_{K(i)}$ for every i, such that $e_{K(i)}$ occurs exactly once in the decomposition of e'_i and does not occur in that of $e'_{i'}$ with $i' < i$. Now we have ($n := \#\upsilon$ and $n' := \#\upsilon'$)

$$\begin{aligned}
&\int_{\bar{\mathcal{A}}_\upsilon} f_\upsilon \,\mathrm{d}\mu_{\mathrm{Haar}} \\
&\quad = \int_{\mathbf{G}^n} f_\upsilon(g_1, \ldots, g_n) \,\mathrm{d}\mu_{\mathrm{Haar}} \\
&\quad = \int_{\mathbf{G}^n} f_{\upsilon'} \left(\prod_{k_1} g^{\pm 1}_{j(k_1,1)}, \ldots, \prod_{k_{n'}} g^{\pm 1}_{j(k_{n'},n')} \right) \prod \,\mathrm{d}\mu_{\mathrm{Haar}} \\
&\qquad (f_\upsilon = f_{\upsilon'} \circ \pi^\upsilon_{\upsilon'} \text{ and decomposition of } e'_i) \\
&\quad = \int_{\mathbf{G}} \cdots \int_{\mathbf{G}} f_{\upsilon'}(\cdots g^{\pm 1}_{K(1)} \cdots, \ldots, \cdots g^{\pm 1}_{K(n')} \cdots) \,\mathrm{d}\mu_{\mathrm{Haar},1} \cdots \mathrm{d}\mu_{\mathrm{Haar},n} \\
&\qquad (\text{the dots in } \cdots g^{\pm 1}_{K(l)} \cdots \text{ denote always a product of } g^{\pm 1}_j \text{ with} \\
&\qquad j \neq K(l') \text{ for all } l' > l) \\
&\quad = \int_{\mathbf{G}} \cdots \int_{\mathbf{G}} f_{\upsilon'}(g_1, \ldots, g_{n'}) \,\mathrm{d}\mu_{\mathrm{Haar},1} \cdots \mathrm{d}\mu_{\mathrm{Haar},n'} \\
&\qquad (\text{translation and inversion invariance, normalization of the Haar measure}) \\
&\quad = \int_{\bar{\mathcal{A}}_{\upsilon'}} f_{\upsilon'} \,\mathrm{d}\mu_{\mathrm{Haar}}.
\end{aligned}$$

- F_0 is continuous due to $|F_0(f)| \leq \|f_\upsilon\| = \|f\|$. The last equality follows from the surjectivity of π_υ, see Proposition 3.12.
- F_0 is obviously linear and positive.
- Hence, F is a well-defined, linear, continuous, positive functional on $C(\bar{\mathcal{A}})$.
- Due to the Riesz–Markow theorem there is a unique Borel measure μ_0 on $\bar{\mathcal{A}}$ with $F(f) = \int_{\bar{\mathcal{A}}} f \,\mathrm{d}\mu_0$.
- F is strictly positive.

Let $f \in C(\bar{\mathcal{A}})$, $f \neq 0$, and $k := f^* f \in C(\bar{\mathcal{A}})$. Then $U := k^{-1}((\frac{1}{2}\|k\|, \infty))$ is open and non-empty. Thus, there is a hyph υ and an open, non-empty U_υ with $\pi_\upsilon^{-1}(U_\upsilon) \subseteq U$.

Now we use the fact that every open non-empty subset of a compact Lie group has non-vanishing Haar measure. (In fact, let $V \subseteq \mathbf{G}$ be open, non-empty. Then $\{Vg | g \in \mathbf{G}\}$ is a covering of $\mathbf{G}$. Since $\mathbf{G}$ is compact, there are only finitely many g_i, such that $\cup_{i=1}^n V g_i = \mathbf{G}$. Due to the translation invariance of the Haar measure we have $\mu(V) = (1/n) \sum \mu(V g_i) \geq (1/n)\mu(\mathbf{G}) > 0$.) Consequently, we have

$$\begin{aligned} F(f^* f) &= \int_{\bar{\mathcal{A}}} k \, \mathrm{d}\mu_0 \geq \frac{1}{2} \int_U \|k\| \, \mathrm{d}\mu_0 \\ &\geq \frac{1}{2}\|k\| \int_{\pi_\upsilon^{-1}(U_\upsilon)} 1 \, \mathrm{d}\mu_0 = \frac{1}{2}\|k\| \int_{U_\upsilon} 1 \, \mathrm{d}\mu_{\mathrm{Haar}} = \frac{1}{2}\|k\| \mu_{\mathrm{Haar}}(U_\upsilon) > 0. \qquad \square \end{aligned}$$

7. Discussion

In this paper we investigated for some examples how the theory of generalized connections depends on the chosen smoothness category for the paths used in the construction of $\bar{\mathcal{A}}$. The most important theorem yields that in every case an induced Haar measure can be defined. But, there are some problems that depend very crucially on the smoothness of the paths. So let us resume the discussion of the beginning of this paper: What could be a good choice of smoothness conditions?

One decisive point is the denseness of the classical (smooth) connections in the space $\bar{\mathcal{A}}_{(r)}$. In the case of compact structure groups $\mathbf{G}$ the denseness has been proven for the immersive smooth [5,10] and piecewise analytic category [11]. However, in the first case [5] the space $\bar{\mathcal{A}}_{\mathrm{web}}$ was defined not by $\varprojlim_w \bar{\mathcal{A}}_w$, but by $\varprojlim_w \bar{\mathcal{A}}_w$ where $\mathcal{A}_w$ (being a Lie subgroup of $\mathbf{G}^{\# w}$) denotes the image of the space $\mathcal{A}$ of *regular* connections under the map $\pi_w \equiv h_{c_1} \times \cdots \times h_{c_W}$. Thus, the denseness follows immediately by the directedness of the set of webs (cf. Appendix B). Supposed, $\mathbf{G}$ is in addition semi-simple, Lewandowski and Thiemann [10] proved that $\mathcal{A}_w = \bar{\mathcal{A}}_w = \mathbf{G}^{\# w}$ which implies that $\mathcal{A}$ is also dense in our $\bar{\mathcal{A}}_{(\infty,+)}$. Up to now, we do not know whether this is true for arbitrary Lie groups. However, $\mathcal{A}$ is definitely *not* dense in the space $\bar{\mathcal{A}}_{(r)}$ for non-immersed paths. Let, e.g., γ be an immersed path without self-intersections and $\gamma'(\tau) := \gamma(\tau^2)$. Then γ' is not equivalent to γ (cf. [6]) and not an immersion. But, obviously $h_\gamma(A) = h_{\gamma'}(A)$ for all $A \in \mathcal{A}$. Consider now two elements $g, g' \in \mathbf{G}$ and corresponding disjoint open neighborhoods $U, U' \subseteq \mathbf{G}$. We see that $\upsilon := \{\gamma, \gamma'\}$ is a hyph and so $\pi_\gamma^{-1}(U) \cap \pi_{\gamma'}^{-1}(U') = \pi_\upsilon^{-1}(U \times U')$ is non-empty and open, but contains no regular A. So $\mathcal{A}$ is not dense in $\bar{\mathcal{A}}_{(r)}$.

Since this is, in fact, very unsatisfactory, we should look for other possibilities for the definition of the set $\mathcal{P}$ for non-immersive paths. The probably easiest way should be to redefine the equivalence relation between paths. Why should non-self-intersecting paths γ and γ' only be equivalent if they coincide up to a piecewise C^r-transformation? Perhaps we should use a definition of the following kind: $\gamma \sim \gamma'$ iff $h_A(\gamma) = h_A(\gamma')$ for all $A \in \mathcal{A}$—maybe at least provided $\mathrm{im}\,\gamma = \mathrm{im}\,\gamma'$. This one is quite similar to that used originally in [1,2]. On the one hand, we expect that all the constructions made in this paper and its campanion [6] will still go through. But, on the other hand, even for that definition we do not see that it saves the desired density property in more cases than described above.

What other questions discussed in the Ashtekar framework could be touched by the choice of $\mathcal{P}$? One area we mentioned above — the diffeomorphism invariance of quantum gravity. Here, obviously, we have to admit at least smooth paths. Another problem is quantum geometry. For instance, the definition of the area operator [4] enforced the usage of at most the analytic category. There one has to calculate sums over intersection points of spin networks with surfaces. But, since there can exist infinitely many such points when working with smooth paths, these sums can be infinite. This problem could be solved if there would exist for every fixed surface S in M a basis of $L_2(\bar{\mathcal{A}}, \mu_0)$, such that every base element has only finitely many intersection points with S. But this seems very unlikely.

Acknowledgements

The author was supported by the Max-Planck-Institut für Mathematik in den Naturwissenschaften in Leipzig.

Appendix A. Additional results for $\bar{\mathcal{A}}/\bar{\mathcal{G}}$

In this appendix we give three corollaries about assertions that can be proven not only for $\bar{\mathcal{A}}$, but also for $\bar{\mathcal{A}}/\bar{\mathcal{G}}$. For the definition of $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ and the used notation we refer to [6].

Corollary A.1. *$\pi_\Gamma : \overline{\mathcal{A}/\mathcal{G}} \to \overline{\mathcal{A}/\mathcal{G}}_\Gamma$ and $\pi_\Gamma : \bar{\mathcal{A}}/\bar{\mathcal{G}} \to \overline{\mathcal{A}/\mathcal{G}}_\Gamma$ are surjective for all graphs Γ.*

Proof. Let $[h_\Gamma] \in \overline{\mathcal{A}/\mathcal{G}}_\Gamma \equiv \bar{\mathcal{A}}_\Gamma/\bar{\mathcal{G}}_\Gamma$. From Proposition 3.12 follows the existence of an $h \in \bar{\mathcal{A}}$ with $\pi_\Gamma(h) = h_\Gamma$. Then, $([\pi_{\Gamma'}(h)])_{\Gamma'} \in \overline{\mathcal{A}/\mathcal{G}}$ with $\pi_\Gamma(([\pi_{\Gamma'}(h)])_{\Gamma'}) = [\pi_\Gamma(h)] = [h_\Gamma]$. Analogously $\pi_\Gamma([h]) = [h_\Gamma]$ holds for $[h] := \pi_{\bar{\mathcal{A}}/\bar{\mathcal{G}}}(h) \in \bar{\mathcal{A}}/\bar{\mathcal{G}}$, whereas $\pi_{\bar{\mathcal{A}}/\bar{\mathcal{G}}} : \bar{\mathcal{A}} \to \bar{\mathcal{A}}/\bar{\mathcal{G}}$ is the canonical projection. □

Corollary A.2. *$\pi_\Gamma : \bar{\mathcal{A}}/\bar{\mathcal{G}} \to \bar{\mathcal{A}}_\Gamma/\bar{\mathcal{G}}_\Gamma \equiv \overline{\mathcal{A}/\mathcal{G}}_\Gamma$ is open for all graphs Γ.*

Proof. This assertion comes from the surjectivity and the continuity of $\pi_{\bar{\mathcal{A}}/\bar{\mathcal{G}}}$, from the openness of $\pi_\Gamma : \bar{\mathcal{A}} \to \bar{\mathcal{A}}_\Gamma$ and $\pi_{\bar{\mathcal{A}}_\Gamma/\bar{\mathcal{G}}_\Gamma}$ as well as from the commutativity of the following diagram:

$$\begin{array}{ccc} \overline{\mathcal{A}} & \xrightarrow{\pi_{\overline{\mathcal{A}}/\overline{\mathcal{G}}}} & \overline{\mathcal{A}}/\overline{\mathcal{G}} \\ \Big\downarrow{\scriptstyle \pi_\Gamma} & & \Big\downarrow{\scriptstyle \pi_\Gamma} \\ \overline{\mathcal{A}}_\Gamma & \xrightarrow{\pi_{\overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma}} & \overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma \end{array} .$$

□

Every measure on a compact $\bar{\mathcal{A}}$ induces a measure on $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ via

Definition A.3. Let μ be a Borel measure on $\bar{\mathcal{A}}$.
Define $\mu_{\bar{\mathcal{G}}}(U) := \mu(\pi^{-1}_{\bar{\mathcal{A}}/\bar{\mathcal{G}}}(U))$ for all Borel sets U on $\bar{\mathcal{A}}/\bar{\mathcal{G}}$.

Proposition A.4. *$\mu_{\bar{\mathcal{G}}}$ is a Borel measure on $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ for all Borel measures μ on $\bar{\mathcal{A}}$.*

Especially, the induced Haar measure can be transferred from $\bar{\mathcal{A}}$ to $\bar{\mathcal{A}}/\bar{\mathcal{G}}$.

Appendix B. Denseness lemma for projective limits

Lemma B.1. *Let A be a set, X_a be a topological space for each $a \in A$ and "$\leq$" be a partial ordering on A. Let $\pi^{a_2}_{a_1} : X_{a_2} \to X_{a_1}$ for all $a_1 \leq a_2$ be a continuous and surjective map with $\pi^{a_2}_{a_1} \circ \pi^{a_3}_{a_2} = \pi^{a_3}_{a_1}$ if $a_1 \leq a_2 \leq a_3$. Furthermore, let $\pi_a : \varprojlim_{a' \in A} X_{a'} \to X_a$ be the usual projection on the a-component and X be some subset of $\varprojlim_{a \in A} X_a$.*

Then X is dense in $\varprojlim_{a \in A} X_a$ if

1. *A is directed, i.e., for any two $a', a'' \in A$ there is an $a \in A$ with $a', a'' \leq a$, and*
2. *$\pi_a(X)$ is dense in X_a for all $a \in A$.*

Proof. Let $U \subseteq \varprojlim_a X_a$ be open and non-empty, i.e., $U \supseteq \cap_i \pi^{-1}_{a_i}(V_i) \neq \emptyset$ with open $V_i \subseteq X_{a_i}$ and finitely many $a_i \in A$. Since A is directed, there is an $a \in A$ with $a_i \leq a$ for all i and thus $U \supseteq \pi^{-1}_a(\cap_i (\pi^a_{a_i})^{-1}(V_i))$ with non-empty $V := \cap_i (\pi^a_{a_i})^{-1}(V_i) \subseteq X_a$. V is open because $\pi^a_{a_i}$ is continuous. Since $\pi_a(X)$ is dense in X_a for all a, there is an $x \in X$ with $\pi_a(x) \in V$ and so $\pi_{a_i}(x) \in V_i$ for all i, hence $x \in U$. $\square$

References

[1] A. Ashtekar, C.J. Isham, Representations of the holonomy algebras of gravity and nonabelian gauge theories, Classical Quantum Gravity 9 (1992) 1433–1468.

[2] A. Ashtekar, J. Lewandowski, Representation theory of analytic holonomy C^* algebras, in: J.C. Baez (Ed.), Knots and Quantum Gravity, Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 1994.

[3] A. Ashtekar, J. Lewandowski, Projective techniques and functional integration for gauge theories, J. Math. Phys. 36 (1995) 2170–2191.

[4] A. Ashtekar, J. Lewandowski, Quantum theory of geometry. I: Area operators, Classical Quantum Gravity 14 (1997) A55–A82.

[5] J.C. Baez, S. Sawin, Functional integration on spaces of connections, J. Funct. Anal. 150 (1997) 1–26.

[6] Ch. Fleischhack, Gauge orbit types for generalized connections, MIS-Preprint 2/2000.

[7] Ch. Fleischhack, Stratification of the generalized gauge orbit space, MIS-Preprint 4/2000.

[8] Ch. Fleischhack, A new type of loop independence and $SU(N)$ quantum Yang–Mills theory in two dimensions, J. Math. Phys. 41 (2000) 76–102.

[9] J.L. Kelley, General Topology, Van Nostrand, New York, 1955.

[10] J. Lewandowski, T. Thiemann, Diffeomorphism invariant quantum field theories of connections in terms of webs, Classical Quantum Gravity 16 (1999) 2299–2322.

[11] A.D. Rendall, Comment on a paper of Ashtekar and Isham, Classical Quantum Gravity 10 (1993) 605–608.